Percolation meets Quantum ... and some non-perturbative results follow

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70 Years of Percolation
Cambridge Univ.
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Some celebrated examples:

$*$ 2D Ising (Onsager, Kaufmann, ... Schultz-Mattis-Lieb, Kadanoff, ...)

$*$ 6 vertex models ... (Yang, Lieb, Baxter, ...)

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- Ground states of quantum d-dimensional systems $\Longleftrightarrow$
thermal states of classical $d+1$-dimensional systems.

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A general relation, that will be recalled here,

- Ground states of quantum d-dimensional systems $\Longleftrightarrow$
  thermal states of classical $d+1$-dimensional systems. Consequently:

- Rather different looking models, some classical and some quantum,
  with different forms of symmetry breaking, share a *common mathematical scaffolding*.

- And it seems that the simplest way to explain each is by combining the different perspectives.

A key role in the above is played by a long-recognized common dichotomy associated with $2D$ loop-soup models.

A random cluster system that is an extension of a continuous-time percolation model is naturally expressed in terms of a 1+1 dimensional loop-soup measure, whose loops form the inner or outer boundaries of the model's connected clusters.

The same random loop system shows up in a stochastic geometric representation of three distinct quantum spin models.

Combining insights based on by the different "projections" of the common loop models one arrives at a structural explanation of threshold parameter values for three very different phenomena:

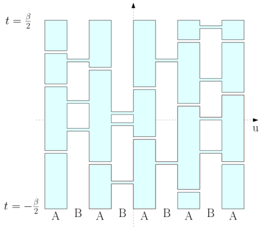
i) discontinuity of the phase transition in a planar classical random cluster model or equivalently: degeneracy and symmetry breaking in the ground state(s) of an infinite self-dual quantum Q-state Potts spin chain, at $\boxed{Q > 4}$.

iii) dimerization of the ground states of a flattened version of the quantum Heisenberg model at $\boxed{S > 1/2}$,

iv) Nèel order in the ground state of the a-symmetric H_{XXZ} quantum spin chain at $\boxed{\Delta > 0}$.

(Talk based on a joint 2020 paper with H. Duminil-Copin and S. Warzel, with input from previous works by G. Ray and Y. Spinka (2020) on Q state Potts models, and Aizenman and Nachtergaele (1994) on quantum spin chains.)

Starting from the partition of $\mathbb{R}^2$ into A/B stripes, consider the continuum percolation model in which randomly placed "edges" over A strips serve as B-connecting paths disrupting A connection, and vice versa, at edge Poisson densities (λ_A, λ_B)



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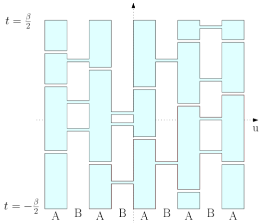

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$$\boxed{Z(L_{hor}, L_{vert}) = \int Q^{N_A(\omega)} \rho_{\lambda_A, \lambda_B}(d\omega) \approx \int \sqrt{Q}^{N_\ell(\omega)} \rho_{\lambda_A\sqrt{Q},\, \lambda_B/\sqrt{Q}}(d\omega)}$$

where $\boxed{N_A(\omega) \;=\; \#\{\text{connected } A\text{-clusters}\}}$ and $\boxed{N_\ell \;=\; N_A + N_B}$

When $\lambda_A\sqrt{Q} = \lambda_B/\sqrt{Q}$ the model is self dual(!)

An interesting question: under what conditions will this symmetry be broken?

Note: the duality symmetry coincides here with shift invariance !

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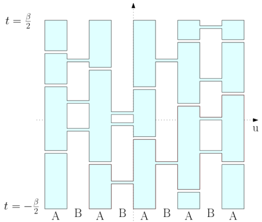

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For those familiar with the similar question for Q state Potts model*, it will be natural to expect the transition to occur at $\boxed{Q_c = 4}$. *And indeed that is so* [ADW].

(* [Yang, Baxter, Lieb,...,Duminil-Gagnebin-Harel-Manolescu-Tassion '16, Ray-Spinka'19,...])

Ising spin chain (on $\mathbb{Z}$) with a transverse magnetic field ($Q = 2$)

Single ‘q-bit’:

Hilbert space $\mathcal{H}^{(2)} = \text{span}\{|+\rangle, |-\rangle\}$.

Pauli spin operators: $\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

The Hilbert space for a quantum spin chain: $\mathcal{H} = \bigotimes_{j \in \mathbb{Z}} \mathcal{H}_j^{(2)}$

The Hamiltonian (acting on $\mathcal{H}$): $\boxed{H = -\sum_j \left[\sigma_j^z \sigma_{j+1}^z + \sigma_j^x\right]}$ with $R_j \equiv \mathbb{1} \otimes \ldots \mathbb{1} \otimes R \otimes \mathbb{1} \ldots \otimes \mathbb{1}$

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The Q-valued quantum Potts model:

$$\sigma_j^z \quad \Longrightarrow \text{the diagonal matrix } D_j = \begin{pmatrix} 1 & 0 & 0\ldots \\ 0 & 2 & 0\ldots \\ 0 & 0 & 3\ldots \end{pmatrix}$$

$$\sigma_j^x \quad \Longrightarrow \text{the corresponding flip operator,} \quad \text{and} \quad \sigma_j^z \sigma_{j+1}^z \quad \Longrightarrow \delta_{D_j, D_{j+1}}$$

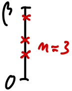
The quantum model's $\boxed{\mathrm{tr}\, e^{-\beta H_{Q,L}}}$ coincides with the partition function of the above continuum $1+1$ dimensional classical Potts model in $[0, \beta] \times [0, L]$.

For $\boxed{\beta >> 1}$ the classical Gibbs state yields the quantum system's low temperature dependence, and in the limit $\beta \to \infty$ its ground state(s)

Feynman, Dyson, Ginibre '71, Suzuki-Trotter '76, .., Aiz.-Lieb '90, Conlon-Solovej '91, Toth '93, Aiz.- Nacht. '94.,...
Aiz., Duminil-Copin Warzel '20, ... Björnberg, Mühlbacher, Nachtergaele, Ueltschi '21

Warmup: $\boxed{e^{\beta(K-1)} = \sum_{n=0}^{\infty} p_n K^n \equiv \mathbb{E}(K^n)}$ with $p_n = \frac{\beta^n}{n!} e^{-\beta}$

(the Poisson distribution)



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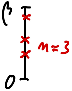
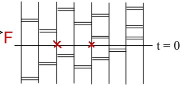
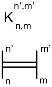


$$\boxed{e^{\beta \sum_{b\in\mathcal{E}(\Lambda)}(K_b-1)} = \int_{\Omega(\Lambda,\beta)} \rho(d\omega)\ \mathcal{T}\left(\prod_{(b,t)\in\omega} K(b,t)\right)}$$

$\omega \Leftrightarrow$



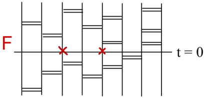


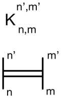
$\Omega(\Lambda,\beta)$ – the set of countable subsets of $\mathcal{E}(\Lambda)\times[0,\beta]$

$\rho(d\omega)$ – the probability measure under which ω forms a Poisson process over Ω, of intensity dt along each "vertical" line $\{b\}\times[0,\beta]$.

One gets:

$$\mathrm{tr}\, e^{-\beta H/2} F e^{-\beta H/2} = \int_{\Omega(\Lambda,\beta)} \rho(d\omega)\, \mathrm{tr}\ \mathcal{T}$$





E.g., for quantum spin models thermal expectation values may be expressed in terms of an integral over histories of $\{S^z_x\}$ (in "imaginary time"), i.e. configurations of $\sigma^3(x,t)$ defined over $[-L_1, L_2], \times[0,\beta]$.

Each quantum operator F (acting on the Hilbert space associated with Λ) is represented by a specific action on this functional integral (typically at $t=0$).

Now apply this to a chain of spin S quantum spins with $\boxed{H_{AF} = -\sum(2S+1)P^{(0)}_{u,u+1}}$ (Barber-Batchelor '89

which in case $S = 1/2$ coincides with the standard Heisenberg anti-ferromagnet. Affleck '90, Klumper '90.)

In the basis of e.funct's of $\{S^z_u\}$: $\boxed{(2S+1)P^{(0)}_{u,v} = \sum_{m,m'=-S}^{S} (-1)^{m-m'} \, |m, -m\rangle\langle m', -m'|}$.

In this case, the signs can be gauged away (!) through $U = e^{i\pi\eta/2}$ at $\eta = \sum_u (-1)^u S^z_u$.

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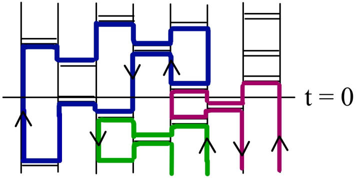
The above approach yields a stochastic geometric representation of the thermal states in terms of a system of random loops, along each of which $S^z(u,t)$ is restricted to $\pm m$ at a constant $m \in [-S, S]$, with $\pm$ flipping upon each "time reversal" (AN94). In this represenation:



$$\boxed{e^{-\beta H_{AF}} = e^{\beta|\mathcal{E}(\Lambda)|} \int_\Omega \rho(d\omega) \prod_j^{*} K_{(b_j,t_j)}}$$

$$\mathrm{tr}\, \mathcal{T}\left(\prod_{(b,t)\in\omega} K(b,t) \right) = (2S+1)^{N_1(\omega)}$$

$$\therefore \boxed{\mathrm{tr}\, e^{-\beta H_{AF}} = \int_{\Omega(\Lambda,\beta)} \rho(d\omega)\ (2S+1)^{N_1(\omega)}}$$

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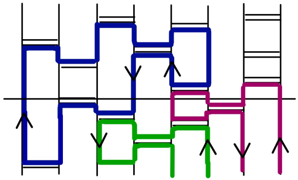

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$$\therefore \boxed{\mathrm{tr}\, e^{-\beta H_{AF}} = \int_{\Omega(\Lambda,\beta)} \rho(d\omega)\ (2S+1)^{N_1(\omega)}}$$

and the thermal spin correlations can be presented as: $\boxed{\langle\sigma^z_u\sigma^z_v\rangle_\beta = (-1)^{|u-v|} C_S \,\mathrm{Pr}(u \overset{\omega}{\longleftrightarrow} v)}$

Note the similarity with the self-dual Qrandom cluster model with $\boxed{(2S+1) \longleftrightarrow \sqrt{Q}}$.

Theorem 1 (AN'94) In the infinite-volume limit, any dense periodic loop-soup measure, with a local finite-energy condition, exhibits
either slow decay of correlations (connectivity) or long range order.
More explicitly: either

$$\boxed{\sum_{u \in \mathbb{L}_1} |x|\, \tau(0,x) = \infty}$$

or else there exists a bounded measurable function $m(\omega)$ with

$$\boxed{\mathbb{E}(T_x \omega)) = (-1)^x}.$$

Key idea: If the loop configuration includes (a.s.) ∞ set of nested loops then option 1. Otherwise option 2.

Theorem 2 In the following models (the first classical the other two quantum)

$$\begin{cases} \text{the classical } Q \text{ state Potts model at } \beta_c \\ \text{for the ground states of } H_{AF} \\ \text{for the ground states of } H_{XXZ} \text{ at } S=\frac{1}{2} \end{cases} \quad \text{LRO (option 2) holds exactly for} \begin{cases} Q > 4 \\ S > 1/2 \\ \Delta > 1 \end{cases}$$

$H_{\text{Potts}} = \sum_u \delta_{\sigma_u, \sigma_{u+1}}$ option (2) $\Leftrightarrow$ discontinuity in the spontaneous magnetization

$$H_{AF} = -\sum_{u=-L+1}^{L-1} P^{(0)}_{u,u+1}$$ option (2) $\Leftrightarrow$ dimerization

$$H^{S=1/2}_{XXZ} = \sum_{u=-L+1}^{L-1} \left[\underline{\sigma}_u \cdot \underline{\sigma}_{u+1} + (\Delta - 1)\sigma^z_u \sigma^z_{u+1}\right];$$ option (2) $\Leftrightarrow$ Ne'el order (staggered magnetization)

What makes $\sqrt{Q} = (2S+1) = 2$ into a threshold value?

Two tracks to the answer (developed initially in the context of Q state Potts models):

I) Bethe ansatz analysis of the 6-vertex models (extended to handle also the model's 1–directional continuum limit) (carried rigorously in *Duminil-Copin, Gagnebin, Harel, Manolescu, and Tassion '16).*

II) An old hint (Baxter–Kelland–Wu '78):

Writing $\boxed{\sqrt{Q} = e^{\lambda} + e^{-\lambda}}$ $\left[\text{ or correspondingly } (2S+1) = e^{\lambda} + e^{-\lambda}\right]$ $(*)$

allows to express the factor $\boxed{\sqrt{Q}^{N_1(\omega)} = (e^{\lambda} + e^{-\lambda})^{N_1(\omega)}}$ as a product of "local action" terms.

BKW noted that the solution of $(*)$ for λ changes its nature (from imaginary to real) at $\sqrt{Q} = 2$, and proposed that this should be significant.

The challenge to develop an argument based on (II) was finally met in G. Ray, Y. Spinka *A short proof of the discontinuity of phase transition in the planar random-cluster model with $q > 4$*, (CMP 2020).

A key role in the Ray-Spinka argument was played by a related random height function.

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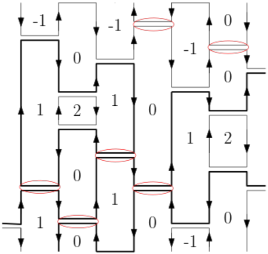
- In our joint work with Hugo Duminil-Copin and Simone Warzel a somewhat analogous argument was developed for the quantum spin chains with H_{AF}.
- The analysis revealed and utilized an emergent relation with the H_{XXZ} spin models at $S = 1/2$. (Coincidence in their spectra was noted by other means before.)

The height function associated with a specified configuration of rungs and loop orientations, and the related binary pseudo spin (defined by the arrows).



Note: vertical discontinuity in the pseudo spin τ implies the presence of a rang of ω.
However, ω also includes other rungs (marked in small ovals) whose presence is transparent to τ. Their knowledge is essential for the full reconstruction of the loops of ω, but not for the height function.

Lemma 1: If for a given $Q > 4$, and λ satisfying $\sqrt{Q} = e^{\lambda} + e^{-\lambda}$ the loop's limiting measure is translation invariant then: i) its typical configurations include an infinite family of nested loops
ii) the distribution of the corresponding height function is not an even function of λ.

(Proof based on percolation analysis, as in Duminil-Gagnebin-Harel-Manolescu-Tassion '16)

Lemma 2: Under the above assumption, tor real λ the distribution of the corresponding height function is an even function of λ.
(Proof idea: the hight function can be determined from just the pseudo spin τ (it does not require the full loop information). But the distribution of τ is that of the spin 1/2 chain under the Hamiltonian H_{XXZ} at $\Delta = \cosh(\lambda)$.)

The combination of these two properties allows to rule out delocalization for the height function at $Q > 4$, and by implications to prove symmetry breaking / LRO at the corresponding values of Q / S / Δ.

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Cf. also the talk by Jacob Björnberg earlier in this meeting.

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Hugo Duminil-Copin
Geneva Univ.

Matan Harel
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Thank you for your attention!